On the use of Lagrangian Coherent Structures in direct assimilation of ocean tracer images

O. Titaud*, J. Verron**, J.-M. Brankart**
titaud@cerfacs.fr

CERFACS*, Toulouse, France
LEGI**, Grenoble, France

GlobCurrent 2012
Ifremer, Brest, France, 7-9 March 2012
Objectives of the study

▶ The main objective of this study is to show that we can exploit ocean tracer images in direct image assimilation schemes.

▶ We realize a numerical experiment using a high resolution double-gyre idealized model of the North Atlantic Ocean (1/54°).

▶ We will focus on:
  ▶ Surface velocity fields
  ▶ Sea Surface Temperature (SST)
  ▶ mixed layer phytoplankton (PHY)

▶ We construct an observation operator based on the computation of Lagrangian Coherent Structures.

▶ We study the sensibility of a cost function associated with this operator wrt the amplitude of a surface velocity perturbation (state variable).
Outline

Direct Image Assimilation

Test case

Coherent Lagrangian Structures
  Definition of Finite-Time Lyapunov Vectors

Observation operators based on LCS computation
  Observation operator based on FTLV

Impact study
  Methodology
  Results

Conclusions, future work, references
Direct Image Assimilation

General concept

- **Ocean tracer images** contain structured information that should be exploited

- **$S$: space of pertinent information** to be observed: **structures**
  - Frequency characteristics (e.g. multi-scale modelling of the images)
  - Pattern properties (contours, regions of interest . . .)

- **$\| \cdot \|_S$: discrepancy measure** between two elements of $S$

- **$\mathcal{H}_S$: structures observation operators** (model equivalent of obs structures)

\[
J(X_0) = \frac{1}{2} \int_0^T \| \mathcal{H}[X] - y_{obs} \|^2_0 dt + \frac{1}{2} \int_0^T \| \mathcal{H}_S[X] - y_s \|^2_S dt + \frac{1}{2} \| x_0 - x_b \|^2_X
\]

  - classical term
  - "image" term

- **$y \in S$: observed structures in images** (sub-sampling of observations)

Titaud et al., 2010
Direct Image Assimilation

**Test case**

Coherent Lagrangian Structures

*Definition of Finite-Time Lyapunov Vectors*

Observation operators based on LCS computation

*Observation operator based on FTLV*

**Impact study**

Methodology

Results

Conclusions, future work, references
Test case

High resolution (1/54°) idealized simulation of the North Atlantic Ocean (double gyre)

- NEMO-OPA/TOP2 (dynamics/tracers) and LOBSTER (bio-geochemical)
- Sea Surface Temperature (SST) and mixed layer phytoplankton (PHY)
- Region of study: Ω = [−74.62, −68.62] × [22.36, 28.36] (6° × 6°)
- Reference date: April 9

Sequence of meso-scale surface velocities (1/4°) obtained by sub-sampling and spatial filtering (Lanczos)
Direct Image Assimilation

Test case

Coherent Lagrangian Structures

**Definition of Finite-Time Lyapunov Vectors**

Observation operators based on LCS computation

Observation operator based on FTLV

Impact study

Methodology

Results

Conclusions, future work, references
Coherent Lagrangian Structures (LCS)

The transport of a tracer in a fluid is closely related to emergent patterns called Coherent Structures (Ottino 1989, Wiggins 1992):

- Stationary flows: stable and unstable manifolds of hyperbolic trajectories
- Delimit regions of whirls, stretching or contraction

In practice, LCS are determined by computing the Finite Time Lyapunov Exponents (FTLE)

(Haller and Yuan, 2000), (Haller, 2001a; 2001b; 2002; 2011), (Shadden et al., 2005)

This tool is widely used in oceanography to study mixing processes

(d’Ovidio et al., 2004), (Lehahn et al., 2007), (Beron-Vera et al., 2010)
Finite-Time Lyapunov Vectors (FTLV)

\[
\begin{align*}
\frac{Dx(t)}{Dt} &= u(x(t), t) \\
x(t_0) &= x_0 \\
\end{align*}
\]

Particle transport by the flow \(u(x, t)\)

\[
\begin{align*}
\frac{D\delta x(t)}{Dt} &= \nabla u(x(t), t) \cdot \delta x(t) \\
\delta x(t_0) &= \delta_0, \quad x(t_0) = x_0
\end{align*}
\]

Evolution of a given perturbation \(\delta x\)

**Finite-Time Lyapunov Vector**

FTLV is defined as the direction of maximum stretching, i.e. the eigenvector \(\phi^{t_0+T}_{t_0}(x_0)\) of the largest eigenvalue \(\lambda_{\text{max}}\) of the Cauchy-Green strain tensor:

\[
\Delta = \left[ \nabla \phi^{t_0+T}_{t_0}(x_0) \right]^* \left[ \nabla \phi^{t_0+T}_{t_0}(x_0) \right], \quad \phi^{t_0+T}_{t_0} : x_0 \mapsto x(T), \quad \text{flow map of (⋆)}
\]

- **Backward FTLV** (≈ stable manifold): time integration is inverted in (⋆)

- **Finite-Time Lyapunov Exponent** (separation rate): \(\frac{1}{|T|} \ln \sqrt{\lambda_{\text{max}}(\Delta)}\)

(Ott, 1993), (Shadden et al., 2005; 2009), (Haller, 2011)
FTLV: variational point of view

- **FTLV is a local notion**: the eigenvector \( \varphi_{t_0+T} \) is computed at a given point \( x_0 \)
- Seeding a domain with particles initially located on a grid leads to the computation of a discretized vector field

---

\[ \Phi[u] : x \in \Omega \rightarrow \varphi_{t_0+T}(x) \in \mathbb{R}^2 \]
Direct Image Assimilation

Test case

Coherent Lagrangian Structures
  Definition of Finite-Time Lyapunov Vectors

Observation operators based on LCS computation
  Observation operator based on FTLV

Impact study
  Methodology
  Results

Conclusions, future work, references
Connection between FTLV and tracer fields

The orientation of the gradient of passive tracers converge to that of backward FTLV in freely decaying 2D turbulence flow

(Lapeyre, 2002)

This property has also been observed on real data

(d’Ovidio et al., 2009)
Observation Operator based on FTLV

- **Structure Space**: functions with values in the Euclidean sphere $S^2$

  $$S = \{ f : \Omega \to S^2 \}$$

- **Observation Operator** (vector field)

  $$\mathcal{H}_S(X) = \Phi(u) \quad \Phi(u) : x \in \Omega \mapsto \varphi_0^{-T}(x) \in S^2$$

- **Information extraction** from the observed image $c$ (vector field)

  $$V : \mathcal{I}_\Omega \to S \quad V(c)(i,j) = \frac{\nabla c(i,j)}{\|\nabla c(i,j)\|} = y \in S^2$$

- Orientation of $v = (u, v) \in S^2 : \Theta(v) = \text{atan}(v) \in [-\pi/2, \pi/2]$

- **Angular measure** in $S$

  $$\|f - g\|_S = \sqrt{\frac{1}{n \times m} \sum_{i,j} \sin^2[\Theta(f(i,j)) - \Theta(g(i,j))]}$$

**Corresponding image part of the cost function**

$$J_S(u) = \|\Phi(u) - V(c)\|_S^2.$$
Direct Image Assimilation

Test case

Coherent Lagrangian Structures
  Definition of Finite-Time Lyapunov Vectors

Observation operators based on LCS computation
  Observation operator based on FTLV

Impact study
  Methodology
  Results

Conclusions, future work, references
Methodology: Pre-requisite for data assimilation

**Aim:** study the behaviour of the cost function with respect to the amplitude \( \lambda \) of velocity perturbations on the form \( u_0 + \lambda \delta u \), where \( \delta u \sim \mathcal{N}(0, SS^T) \)

We perturb the initial field of the sequence \( (u_k)_{k=-10}^{k=0} \):

\[
\begin{align*}
    u_k^\lambda &= \begin{cases} 
    u_0 + \lambda \delta u & \text{if } k = 0 \\
    u_k & \text{else}
    \end{cases} \\
    u^\lambda &= (u_k^\lambda)_{k=-10}^{k=0}
\end{align*}
\]

\[
\mathcal{H}_S[u^\lambda] = \text{FTLV} = \varphi_0^T(u^\lambda)
\]

\[
\downarrow -T
\]

\[
\begin{array}{cccc}
    u_{-10} & u_{-9} & \cdots & u_0
    \end{array}
\]

\[
\text{Time}
\]

\[
y = \text{normalized } \nabla \text{tracer}
\]

Sensitivity of the cost function wrt to a perturbation amplitude \( \lambda \):

\[
\tilde{J}_S(\lambda) = \|\mathcal{H}_S[u^\lambda] - y\|^2_S, \quad \lambda \in \Lambda.
\]

- We have to check that the **sensitivity function** \( \tilde{J}_S \) admits a minimum at \( \lambda = 0 \) (**no perturbation**).
Methodology

Climatological covariance matrix for the velocity perturbation

- \((u^{(l)})_{l=1}^{r}\): first \(r = 100\) EOFs of the one year sequence of simulated surface velocity fields

\[
 u_k = \bar{u} + \sum_{l=1}^{m=209} \alpha_k^{(l)} u^{(l)},
\]

- \(S = (u^{(1)}|u^{(2)}| \cdots |u^{(r)})\): reduced rank square root representation of the climatological covariance matrix

\[
 P = \frac{1}{m} \sum_{k=1}^{m+1} (u_k - \bar{u})(u_k - \bar{u})^*
\]

- Gaussian perturbations with zero mean and covariance \(SS^T\)

\[
 \delta u \sim \mathcal{N}(0, SS^T). \quad \delta u = \sum_{l=1}^{r} u^{(l)} \delta x_l \quad \text{with} \quad \delta x_l \sim \mathcal{N}(0,1)
\]

We are interested in perturbations of amplitude \(\lambda\) applied at the reference date:

\[
 u_0 + \lambda \delta u
\]
Direct Image Assimilation

Test case

Coherent Lagrangian Structures
  Definition of Finite-Time Lyapunov Vectors

Observation operators based on LCS computation
  Observation operator based on FTLV

Impact study
  Methodology
  Results

Conclusions, future work, references
Results / discussion

Variation of the sensitivity functions computed wrt the amplitude $\lambda$ of nine random perturbations

- Each of the sensitivity function admits a global minimum
- Minimum is generally reached around $\lambda = 0$ (no perturbations)
- Convex shape: good point for minimization algorithms
- Minimum value is not zero
Direct Image Assimilation

Test case

Coherent Lagrangian Structures
   Definition of Finite-Time Lyapunov Vectors

Observation operators based on LCS computation
   Observation operator based on FTLV

Impact study
   Methodology
   Results

Conclusions, future work, references
Conclusions, future work and references

Conclusions

- **High resolution** ocean tracer images may be exploited by a direct image assimilation scheme in a **mesoscale** model
- **FTLV fields contain information about the system dynamic** that can be observed in the ocean tracer fields: this is a good candidate to construct **observation operator** for image assimilation
- A single ocean tracer image contains a **time integrated information** on the system dynamics

Future work

- Full data assimilation experiment
- Observation errors / real data

References