A particle stochastic filter for fluid flow recovery from images

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Fluminance

Introduction

Geophysical flow analysis

- Great potentiality of satellite images for submesoscale analysis
- Image data are only poorly taken into account in data assimilation

Objective

- Explore stochastic filtering techniques to extract characteristic features of fluid flows along time from image sequences

Goals

- Improve the temporal consistancy of fluid flow velocity measurements
- Deal with innacurate modeling of the large scale dynamics with finer scale image measurements

Stochastic filtering in a non linear setting

Principle

- Given $d\mathbf{x}_t = \mathbf{M}(\mathbf{x}_t)dt + \sigma(t)d\mathbf{B}_t$ and $\mathbf{y}_k = \mathbf{H}(\mathbf{x}_k) + \gamma_k$
- Estimate the pdf $p(\mathbf{x}_{0:k}|\mathbf{y}_{1:k}) = p(\mathbf{x}_{0:k-1}|\mathbf{y}_{1:k-1})\frac{p(\mathbf{y}_k|\mathbf{x}_{t=k})p(\mathbf{x}_{t=k}|\mathbf{x}_{k-1})}{p(\mathbf{y}_k|\mathbf{y}_{1:k-1})}$

Gaussian linear model: Kalman Filtering

$$\mathbb{E}(\mathbf{x}_k|\mathbf{y}_{1:k}) = \mathbf{x}_k^a = \mathbf{x}_{k|k-1} + \mathbf{K}(\mathbf{y}_t - \mathbf{H}\mathbf{x}_{k|k-1}),$$
$$\mathbf{K} = \mathbf{\Sigma}_{k|k-1}\mathbf{H}^T(\mathbf{H}\mathbf{\Sigma}_{k|k-1}\mathbf{H}^T + \mathbf{R})^{-1}$$
$$\mathbb{E}((\mathbf{x}_k - \mathbf{x}_k^a)(\mathbf{x}_k - \mathbf{x}_k^a)^T|\mathbf{y}_{1:k}) = \mathbf{P}_k^a = (\mathbf{I} - \mathbf{K}\mathbf{H})\Sigma_{k|k-1}$$

Non Linear dynamics, linear measure: Ensemble Kalman Filtering

Kalman updates computed from a set of samples $\mathbf{x}_t^{(i)},\ i = 1, \dots, N$

Particle Filter

Non-linear dynamics and observations

- $p(\mathbf{x}_k|\mathbf{z}_{1:k}) \simeq \sum_i w_k^{(i)} \delta_{\mathbf{x}_k^{(i)}}$
- prediction step (importance distribution sampling π)

$$\mathbf{x}_k^{(i)} \sim \pi(\mathbf{x}_{0:k}|\mathbf{y}_{1:k}) = \pi(\mathbf{x}_{0:k-1}|\mathbf{y}_{1:k-1})\pi(\mathbf{x}_k|\mathbf{y}_{1:k}, \mathbf{x}_{0:k-1})$$

- correction step

$$w_k^{(i)} \propto w_{k-1}^{(i)} \frac{p(\mathbf{y}_k|\mathbf{x}_k^{(i)})p(\mathbf{x}_k^{(i)}|\mathbf{x}_{k-1}^{(i)})}{\pi(\mathbf{x}_k^{(i)}|\mathbf{x}_{0:k-1}^{(i)}, \mathbf{y}_{1:k})}$$

Ensemble Kalman filter extension

Importance distribution

- Bootstrap filter

$$\pi(\mathbf{x}_k|\mathbf{x}^{(i)}_{0:k-1}, \mathbf{z}_{1:t}) = p(\mathbf{x}_k|\mathbf{x}^{(i)}_{k-1}) \Rightarrow w^{(i)}_k \propto w^{(i)}_{k-1} p(\mathbf{y}_k|\mathbf{x}^{(i)}_k)$$

$\Rightarrow$ strong limitation in high dimensional space

- Ensemble Kalman proposal distribution

$$\pi(\mathbf{x}_k|\mathbf{x}^{(i)}_{0:k-1}, \mathbf{z}_{1:k}) = p(\mathbf{x}_k|\mathbf{x}^{(i)}_{k-1}, \mathbf{y}_k) \approx \mathcal{N}(\overline{\mathbf{x}}^e_k, (\mathbb{I} - K^e H)\Sigma^e_{k|k-1})$$

$$w^{(i)}_k \propto w^{(i)}_{k-1} \frac{p(\mathbf{y}_k|\mathbf{x}^{(i)}_k) p(\mathbf{x}^{(i)}_k|\mathbf{x}^{(i)}_{k-1})}{\mathcal{N}\left(\mathbf{x}^{(i)}_k - \overline{\mathbf{x}}^e_k; 0, \mathbf{P}^a_k\right)}$$

where with linear observation operator

$$(N-1)\mathbf{P}^a_k = \mathbf{x}^f_k {\mathbf{x}^f_k}^T - \mathbf{x}^f_k {\mathbf{x}^f_k}^T \mathbf{H}^T (\mathbf{H}\mathbf{x}^f_k {\mathbf{x}^f_k}^T \mathbf{H}^T + \tilde{\mathbf{R}})^{-1} \mathbf{H}\mathbf{x}^f_k {\mathbf{x}^f_k}^T$$

Ensemble Kalman filter extension

Importance distribution

- Bootstrap filter

$$\pi(\mathbf{x}_k|\mathbf{x}^{(i)}_{0:k-1},\mathbf{z}_{1:t}) = p(\mathbf{x}_k|\mathbf{x}^{(i)}_{k-1}) \Rightarrow w^{(i)}_k \propto w^{(i)}_{k-1} p(\mathbf{y}_k|\mathbf{x}^{(i)}_k)$$

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$$w^{(i)}_k \propto w^{(i)}_{k-1} \frac{p(\mathbf{y}_k|\mathbf{x}^{(i)}_k) p(\mathbf{x}^{(i)}_k|\mathbf{x}^{(i)}_{k-1})}{\mathcal{N}\left(\mathbf{x}^{(i)}_k - \overline{\mathbf{x}}^e_k; 0, \mathbf{P}^a_k\right)}$$

with nonlinear observation operator

$$(N-1)\mathbf{P}^a_k = \mathbf{x}^f_k {\mathbf{x}^f_k}^T - \mathbf{x}^f_k \mathbf{H}(\mathbf{x}^f_k)^T (\mathbf{H}(\mathbf{x}^f_k)\mathbf{H}(\mathbf{x}^f_k)^T + \tilde{\mathbf{R}})^{-1} \mathbf{H}(\mathbf{x}^f_k) {\mathbf{x}^f_k}^T$$

Vorticity recovering from image data

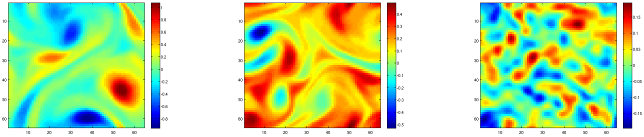
Experiments: 2D velocity-vorticity

- Dynamics

$$d\xi + \nabla\xi \cdot \mathbf{w} dt = \frac{1}{Re}\Delta\xi dt + \sigma_Q dW,$$

- dW isotropic Gaussian field

$$Q(\mathbf{r},\tau) = \mathbb{E}(dW(\mathbf{x},t)dW(\mathbf{x}+\mathbf{r},t+\tau)) = g_\lambda(\mathbf{r})dt\delta(\tau),$$



$\sigma_Q = 0.1$ $\sigma_B = 1$ $\sigma_Q = 0.1$ $\sigma_B = 0.5$ $\sigma_Q = 0.1$ $\sigma_B = 0.1$

Vorticity recovering from image data

Filtering system

- Velocity-vorticity stochastic formulation

$$d\xi + \nabla\xi \cdot \mathbf{w}dt = \frac{1}{Re}\Delta\xi dt + \sigma_Q dW,$$

- Measurements

1) Local motion mesurements

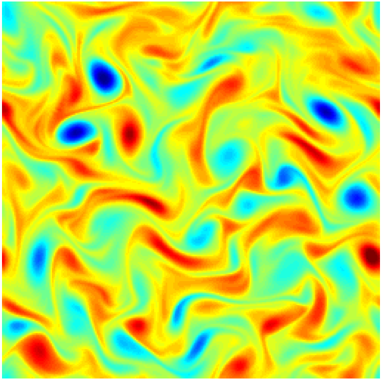
$$\mathbf{y}_k = \mathbf{w} + \gamma_k$$

2) Image reconstruction error

$$\mathbf{I}(\mathbf{x}, k) = \mathbf{I}(\mathbf{x} + \mathbf{d}_{k+1}(\mathbf{x}), k+1) + \boldsymbol{\eta}_k$$

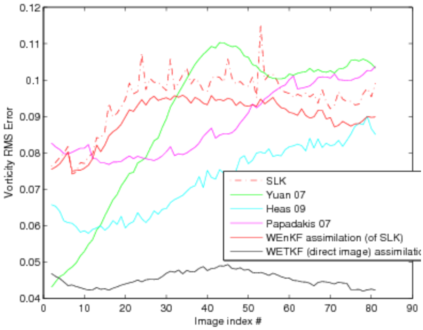
Results: 2D DNS sequence

passive scalar

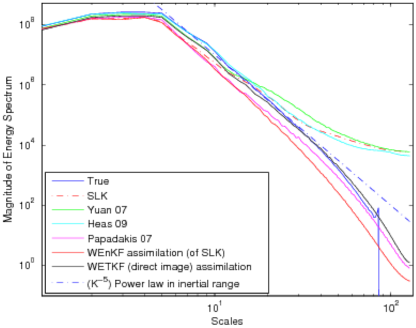


vorticity

Results: 2D DNS sequence



RMSE vorticity



Energy Spectrum

Results: 2D DNS sequence

Filtering results (initialization with local motion estimates)

Results: experimental 2D turbulence on soap film

Soap film thin comb

Results: Oceanic SST images